# Ito Calculus

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1. Processes with bounded variation and quadratic variation for martingales.

<u>Def</u> Let $(\mathcal{F}_t)$ be a filtration. $(A_t)$ is said to have finite variation if it is adapted and $t \to A_t(\omega)$ is right-continuous and has finite variation a.s.

Can define $(X \cdot A)_t = \int_0^t X_s(\omega)\, dA_s(\omega)$ for $X$ such that $X: [0,t] \times \Omega \to \mathbb{R}$ is measurable $(\mathcal{F}_t \times \mathcal{L})$ for any $t$ - progressively measurable (and say, progressively bounded).

But we have: <u>Lemma</u> Continuous martingale of BV is constant.

As in B.M., this would follow from

<u>Thm</u>. If $M$ is a continuous bounded martingale then it has quadratic variation.

Moreover, $\langle M, M\rangle_t$ is unique continuous increasing process such that $M_t^2 - \langle M, M\rangle_t$ - martingale, and $\langle M, M\rangle_0 = 0$.

<u>Pf (outline)</u>: $\forall \Delta$ - partition of $\mathbb{R}_+$,

$$T_t^\Delta := \Sigma (M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_n})^2.$$

$M_t^2 - T_t^\Delta$ - continuous martingale $\Big[ E(M_t^2 - T_t^\Delta | \mathcal{F}_s) = E((M_t^2 - M_s^2)|\mathcal{F}_s) + E(M_s^2 | \mathcal{F}_s) - E(T_t^\Delta - T_s^\Delta | \mathcal{F}_s) = M_s^2 - T_s^\Delta$, since for $t_i < s < t_{i+1}$ $E((M_{t_{i+1}} - M_{t_i})^2 | \mathcal{F}_s) = E((M_{t_{i+1}} - M_s)^2 | \mathcal{F}_s) + (M_s - M_{t_i})^2 \Big]$

Just need to show: $|\Delta_n| \to 0 \Rightarrow T_t^{\Delta_n}$ converges in $L^2_{loc}$, (applying Max inequality) select $\Delta_{n+1}$ to be a refinement of $\Delta_n$.

For any $s, t \in \bigcup \Delta_n$, $\exists n: s, t \in \Delta_n \Rightarrow T_s^{\Delta_n} < T_t^{\Delta_n}$. Since $\bigcup \Delta_n$ dense in $[0,t]$, $\lim T_t^{\Delta_n}$ is an increasing process.

Uniqueness follows from the fact that if $\exists$ another $A_t$, then $\langle M, M\rangle_t - A_t$ is martingale of BV, and thus constant, $= 0$.

<u>Corollary</u> $T$ - stopping time. Then $\underset{\text{stopped martingale}}{\langle M^T, M^T\rangle_t} = \underset{\text{stopped QV}}{\langle M, M\rangle_t^T}$.

<u>Pf</u>. Martingale property.

<u>Problem</u>. Thm does not even cover BM!

2. Local martingales.

<u>Def</u> Local martingale $M_t$ is an adapted process such that $\exists$ sequence of stopping times $T_n$: with

1) $T_n \uparrow$, $\lim T_n = \infty$ a.s.
2) $\forall n$, $X^{T_n}\chi_{\{T_n>0\}}$ is a uniformly integrable martingale.

<u>Remarks:</u> 1) Every martingale is a local martingale ($T_n = n$).

2) Make $S_n = \inf\{t : |X_t| = n\}$. Then $T_n \wedge S_n$ is a stopping time, $T_n \wedge S_n \uparrow \infty$, and $|X_t^{T_n \wedge S_n}| < n$. So, equivalently, may assume it is bounded.

3) Example of local martingale which is not martingale:
$(B_t)$ – standard BM, $T = \inf\{t : B_t = -1\}$. $T < \infty$ a.s.
$B_t^T$ is a martingale, $\lim_{t\to\infty} B_t^T = -1$ a.s.,
$X_t = \begin{cases} B^T_{\frac{t}{1-t}} & 0 \le t < 1 \\ -1 & 1 \le t < \infty \end{cases}$ continuous a.s.

For $t < 1$, $X_t$ is just reparametrized $B_t^T$, and thus martingale so $E(X_t) = E(X_0) = 0$. But for $t \ge 1$, $E(X_t) = -1$, so not a martingale.

Let $T_n = \begin{cases} \inf\{t : X_t = n\}, \\ n, \text{ if } X_t \ne n \ \forall t \in T \end{cases}$ then $T_n \uparrow \infty$, since $\forall \omega$:
$T_n(\omega) = n$ for large $n$, when $n > \max_{0 \le t \le T} B_t$. Thus, a.s. $X_t^{T_n} = B^{T \wedge T_n}_{\frac{t}{1-t}}$ – equal to $-1$
But $X_t^{T_n}$ is a bounded reparametrization of BM, so it is a bounded martingale.

<u>Thm.</u> $M$ – continuous local martingale. Then $\exists!\ \langle M, M\rangle_t$ – continuous, adapted, increasing, $\langle M,M\rangle_0 = 0$ such that
$M_t^2 - \langle M, M\rangle_t$ – continuous local martingale. Moreover, $\forall t\ \forall \Delta_n$ – sequence of partitions of $[0,t]$ with $|\Delta_n| \to 0$,
$\sup_{s \le t} |T_s^{\Delta_n}(M) - \langle M, M\rangle_s| \to 0$ in probability.

<u>Pf.</u> Choose $T_n \uparrow \infty$ stopping times, such that $M_t^{T_n}$ – bounded martingale. $\forall n$, $\exists A_n(t)$, $A_n(0) = 0$, $A_n \uparrow$, $(M_t^{T_n})^2 - A_n(t)$ – martingale.
Observe that $\left((M_t^{T_{n+1}})^2 - A_{n+1}\right)^{T_n}\chi_{\{T_n>0\}}$ is also martingale, equal to $(M_t^{T_n})^2 - A_{n+1}^{T_n}$. By uniqueness, $A_{n+1}^{T_n}(t) = A_n(t)$, $\{T_n > 0\}$.
Set $\langle M, M\rangle_t = A_n(t)$ on $\{T_n > t\}$. Thus $(M^{T_n})^2 - \langle M, M\rangle^{T_n}$ – martingale.
For the second part, fix $\delta, \varepsilon > 0$, $t > 0$. Then $\exists T$ – stopping time

Such that $M^{V}\chi_{\{V>0\}}$ is bounded and $P(V \le t) \le \delta$ (take $T_n$ for large $n$). Since $T^{\Delta}(M)$ and $\langle M, M\rangle$ are the same as $V$-stopped versions on $[0, V]$, we have.

$$P\left(\sup_{s\le t}\left|T^{\Delta}_s(M) - \langle M, M\rangle_s\right| > \varepsilon\right) \le \delta + P\left(\sup_{s\le t}\left|T^{\Delta}_s(M^{V}) - \langle M^V, M^V\rangle_s\right| > \varepsilon\right),$$ and the last term $\to 0$ as $|\Delta| \to 0$ ■

3) Polarization and Kunita-Watanabe inequality.

Polarization: $M, N$ - local continuous martingales. Then $\exists!$ $\langle M, N\rangle_t$ of bounded variation, $=0$ at $0$, such that $MN - \langle M, N\rangle$ - local martingale - bracket of $M$ and $N$.

Pf. $\langle M, N\rangle = \frac{1}{4}\left(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle\right)$.

Uniqueness - from quadratic variation, as before ■

Remarks 1) By uniqueness, if $T$ - stopping time, then
$\langle M^T, N^T\rangle = \langle M, N^T\rangle = \langle M, N\rangle^T$.
($1)=3)$ - obvious, $2)=3)$ follows from the fact that $M^T N^T - M N^T$ is a local martingale).

2) $\langle M, M\rangle = 0 \Leftrightarrow M = \text{const}$ ($M_t = M_0$ a.s. $\forall t$).

Pf. By stopping, enough to consider bounded martingales. Then $E(M_t - M_0)^2 \underset{E(M_t^2 - M_0^2)}{=} E\langle M, M\rangle_t$ ■

An important inequality:

Def. $(H_t)$ is said to be measurable if $(\omega, t) \to H_t(\omega)$ is $\mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (put $\mathcal{F}_\infty$ on $\Omega$, Borel on $\mathbb{R}_+$).

Prop. Let $M, N$ - continuous local martingales, $H, K$ - measurable processes. Then $\forall t \le \infty$ a.s.

$$\int_0^t |H_s||K_s|\,\left|d\langle M, N\rangle_s\right|_{\text{total variation}} \le \left(\int_0^t H_s^2\, d\langle M, M\rangle_s\right)^{\frac{1}{2}} \left(\int_0^t K_s^2\, d\langle N, N\rangle_s\right)^{\frac{1}{2}}$$

Pf. can prove only for $t<\infty$, bdded $H$ and $K$, and take limit. Moreover, by changing sign of $H_s$ and $K_s$, enough to prove that
$$\left|\int_0^t H_s K_s\, d\langle M, N\rangle_s\right| \le \ldots$$

Consider an increasing process $\langle M+rN, M+rN\rangle_t$

Notation: ~~$\langle M, M\rangle$~~ $X^t_s = X_t - X_s$.

Then, for $t>s$ $0 \underset{\text{a.s. } \forall r\text{-rational}}{\le} \langle M+rN, M+rN\rangle^t_s \le \langle M, M\rangle^t_s$ ~~+2r~~

$+ 2r\langle M,N\rangle_s^t + r^2\langle N,N\rangle_s^t$ so $\geq 0\ \forall r\in\mathbb{R}$ by continuity.

so, by computing discriminant, as usual,

$$|\langle M,N\rangle_s^t| \le (\langle M,M\rangle_s^t)^{1/2}(\langle N,N\rangle_s^t)^{1/2}. \quad (*)$$

If we take $k$ of the form

$k = \sum k_i 1_{[t_i,t_{i+1})}$ where $0=t_0<t_1<\dots<t_n=t$ - partition

$H=\sum H_i 1_{[t_i,t_{i+1})}$ then this inequality $(*)$ for finitely many points $(t,s)$ implies

$$\left|\int_0^t H_s k_s \, d\langle M,N\rangle_s\right| \le \sum |H_i k_i| \, |\langle M,N\rangle_{t_i}^{t_{i+1}}| \le$$

$$\le \sum |H_i| (\langle M,M\rangle_{t_i}^{t_{i+1}})^{1/2} \, |k_i| (\langle N,N\rangle_{t_i}^{t_{i+1}})^{1/2} \overset{\text{Cauchy}}{\le}$$

$$\left(\sum H_i^2 \langle M,M\rangle_{t_i}^{t_{i+1}}\right)^{1/2}\left(\sum (k_i)^2\langle N,N\rangle_{t_i}^{t_{i+1}}\right)^{1/2} = \int \times \int,$$

as needed. Then, approximate arbitrary $H,k$ by step-functions.

Corollary. (Kunita – Watanabe inequality).

$\forall p>1$, $\frac1p+\frac1q=1$, we have

$$E\left(\int_0^\infty |H_s||k_s|\,|d\langle M,N\rangle_s|\right) \le \left\|\left(\int_0^\infty H_s^2 \, d\langle M,M\rangle_s\right)^{1/2}\right\|_p \times \left\|\left(\int_0^\infty k_s^2 \, d\langle N,N\rangle_s\right)^{1/2}\right\|_q.$$

Pf. Previous inequality + Hölder

## (4) Semimartingales and Hardy spaces.

Def. A continuous $(\mathcal{F}_t, P)$ - semimartingale is a continuous process $X$ which can be written as $M+A$, where $M$ is a continuous local martingale, $A$ - continuous adapted process of bounded variation.

Properties: 1) $X$ has finite quadratic variation, $\langle X,X\rangle = \langle M,M\rangle$ (because $\langle A,A\rangle=\langle M,A\rangle=0$).

2) Can do polarization $\langle X,Y\rangle$, as before.

$$P\text{-}\lim_{|\Delta|\to 0} \sup_s \left|\sum H_{t_i}(X^s_{t_{i+1}}-X^s_{t_i})(Y^s_{t_{i+1}}-Y^s_{t_i}) - \int_0^s H_u \, d\langle X,Y\rangle_u\right|$$

if $H$ is left-continuous, adapted (for step-functions - the same, as for local martingales, pass to the limit).

Def Hardy space.

[redacted]

Def Hardy space

$\mathbb{H}^2 = \{ M$ - continuous martingale, $\|M\|_{H^2}^2 = \sup E(M_t^2) < \infty \}$. ($\mathbb{H}^2$ - dll/o continuity).

For $M \in \mathbb{H}^2$, by martingale convergence, $\exists M_\infty$: $M_t = E(M_\infty | \mathcal{F}_t)$, and $E(M_\infty^2) = \sup(E(M_t^2))$ (by martingale convergence, again). $\mathbb{H}^2$ - Hilbert space.

Hilbert space $H_0^2 := \{ M \in H^2 : M_0 = 0 \}$. $H^2$ - closed. n.h.

Thm. $M$ - continuous local martingale, $M \in H^2 \iff$

1) $M_0 \in L^2$ and
2) $\langle M, M \rangle$ - integrable; $E(\langle M, M\rangle_\infty) < \infty$

Pf. $T_n \to \infty$ - defining stopping times for $M$. Then $E(M_\infty^2) = \lim_{n\to\infty} E(M_{T_n}^2) = \lim \underbrace{E(M_{T_n}^2 - \langle M, M\rangle^{T_n})}_{\text{martingale}} + \lim_{n\to\infty} E(\langle M, M\rangle^{T_n}) = E(M_0^2) + E(\langle M, M\rangle_\infty)$. if $M \in H^2$. Other direction: use Fatou's lemma to see that $\forall t\ E(M_t^2) \le \underline{\lim}_n E(M_{T_n \wedge t}^2) \le E(M_0^2) + E(\langle M, M\rangle_\infty)$.

Remark. Since $\sup |M_t^2 - \langle M, M\rangle_t| \le \underbrace{(M_\infty^*)^2}_{\text{maximal}} + \langle M, M\rangle_\infty$, we have that for $M \in H^2$, $M_t^2 - \langle M, M\rangle_t$ - uniformly integrable. Remark. $B_t \notin H^2$, but required anywhere while $\mathbb{R}$.

## (5) Stochastic integration.

Def. Let $M \in H^2$. $\mathcal{L}^2(M) = \{ k$ - progressively measurable, [redacted] that $\|k\|_M = E(\int_0^\infty k_s^2 \, d\langle M, M\rangle_s) < \infty \}$.

It is $L^2$ w.r.t. measure on $\mathbb{R}_+ \times \Omega$ given by

$$P_M(\Gamma) = E\left(\int_0^\infty \chi_\Gamma(s, \omega) \, d\langle M, M\rangle_s(\omega)\right).$$

Thm. Let $M \in H^2$, $\forall k \in L^2(M)\ \exists!\ k \cdot M \in H_0^2$:

$$\langle k\cdot M, N\rangle = k \cdot \langle M, N\rangle = \int k \, d\langle M, N\rangle \quad \forall N \in H^2.$$

The map $k \to k \cdot M$ - isometry from $L^2(M)$ into $H_0^2$.

Notation: $(k\cdot M)_t = \int_0^t k_s \, dM_s$ - stochastic (Ito) integral.

Pf. Uniqueness: If $L, L'$ - two martingales in $H_0^2$, and $\forall N \in H^2$

Closedness of $H^2$: $E(\sup |M_t^n - M_t|^2) \underset{\text{Maximal inequality.}}{\le} 4 \|M^n - M\|_{H^2}^2 \to 0$ in measure.
So if $M^n \in H^2$, $M^n \xrightarrow{H^2} M$, then can select unif. conv. subsequence, so $M \in H^2$.

[redacted] [redacted]

$\langle L, N\rangle = \langle L', N\rangle \Rightarrow \langle L-L', L-L'\rangle = 0 \Rightarrow L=0$.

Existence. Assume first $M \in H_0^2$. Then, by Kunita–Watanabe, $\forall N \in H_0^2$ we have

$$\left|E\left[\int k_s \, d\langle M, N\rangle_s\right]\right| \le \|N\|_{H^2} \|k\|_M .$$

so $N \to E(k \cdot \langle M, N\rangle_\infty)$ is continuous in $H_0^2$.

so $\exists\, k\cdot M \in H_0^2$: $\forall N \in H_0^2$ $\quad E((k\cdot M)_\infty N_\infty) = E(k \cdot \langle M, N\rangle_\infty)$.

If $T$ – stopping time, then

$E((k\cdot M)_T N_T) = E(E((k\cdot M)_\infty \mid \mathcal{F}_t) N_t) = E((k\cdot M)_\infty N_T)$.

$E((k\cdot M)_\infty N^T_\infty) = E((k \cdot \langle M, N^T\rangle)_\infty) = E((k \cdot \langle M, N\rangle^T)_\infty) =$

$E((k \cdot \langle M, N\rangle)_T)$. so $(k\cdot M)N - k\cdot\langle M, N\rangle$ – martingale,

so $\langle (k\cdot M), N\rangle = k \cdot \langle M, N\rangle$. Moreover

$\|k\cdot M\|^2_{H^2} = E((k\cdot M)^2_\infty) = E((k^2 \cdot \langle M, M\rangle_\infty) = \|k\|^2_M$ –

isometry.

If $N \in H^2$, $N = N_0 + \tilde{N}$, $\tilde{N} \in H^2$, and $N_0$ does not affect brackets.

If $M \in H^2$, set $k\cdot M := k\cdot(M - M_0)$ – does not affect anything ∎

Properties: 1) If $k = \sum k_i \chi_{[t_i, t_{i+1})}$ – step function, then

$$(k \cdot M)_t = \sum_{i=0}^{n-1} k_i (M_{t_{i+1}} - M_{t_i}) + k_n (M_t - M_{t_n}) \text{ if } t_n \le t < t_{n+1}$$

Using convergence of Riemann sums for $k\cdot\langle M, N\rangle$ and $\langle k\cdot M, N\rangle$, we see that this is indeed the stochastic integral (+ smaller subdivisions $\Delta_n \to 0$).

2) Associativity: $k \in L^2(M)$, $H \in L^2(k\cdot M)$, then $Hk \in L^2(M)$ and $(Hk)\cdot M = H\cdot(k\cdot M)$.

Pf. By associativity of usual integral w.r.t. bdd var ∎

3) If $T$ – stopping time, then

$$k\cdot M^T = k \cdot \chi_{[0,T]} \cdot M = (k\cdot M)^T.$$

Pf $\quad M^T = \chi_{[0,T]} \cdot M$, pair with any $N$:

$$\langle M^T, N\rangle = \langle M, N\rangle^T = \chi_{[0,T]} \cdot \langle M, N\rangle = \langle \chi_{[0,T]} \cdot M, N\rangle.$$

Thus, by associativity

$$k \cdot M^T = (k\,\chi_{[0,T]}) \cdot M \quad \text{and}$$

$$(k\cdot M)^T = \chi_{[0,T]} \cdot (k\cdot M) = (\chi_{[0,T]} k) \cdot M \ ∎$$

(6) Stochastic integral for local martingales.
Idea: BM is square-integrable only on finite intervals, so why not stop it?

Def. $M$ - continuous local martingale.
$L^2_{loc}(M) = \{k$ - progressively measurable, $\exists T_n \uparrow \infty$ - stopping times: $E\left(\int_0^{T_n} k_s^2 \, d\langle M, M\rangle_s\right) < \infty$.
Remark. If $M$ - martingale, $L^2_{loc}(M) = \{k : \int_0^t k_s^2 \, d\langle M,M\rangle_s < \infty \ \forall t$.

Thm. $\forall k \in L^2_{loc}(M)$ $\exists!$ continuous local martingale vanishing at 0, denoted by $k \cdot M = \int k \, dM$ such that $\langle k\cdot M, N\rangle = k \cdot \langle M, N\rangle$ $\forall$ continuous local martingale.
Pf. Choose $T_n \uparrow \infty$; $M^{T_n} \in H^2$, $k^{T_n} \in L^2(M^{T_n})$. $\forall n$; $X_1 = k^{T_n} \cdot M^{T_n}$ - well-defined. By stopping-time property of integrals, $X^{n+1}_t = X^n_t$ as long as $t \le T_n$. So $X_t = X^n_t$ if $t \le T_n$ is well-defined continuous local martingale, and satisfies all the properties.
All the properties of stochastic integral still work.

Def. $k$ - progressively measurable - locally bounded if $\exists T_n \uparrow \infty$ stopping times and some $C_n$, so that $|k^{T_n}| \le C_n$.
Remark. Any continuous process is locally bdd (take $T_n = \inf\{t: |k_t| \ge n\}$).
Locally bounded processes are in $L^2_{loc}(M)$ for every local martingale $M$.
Def. $k$ - loc. bounded, $X = M + A$ - semimartingale, then
$$\int k_s \, dX_s := \underbrace{\int k_s \, dM_s}_{\text{stochastic}} + \underbrace{\int k_s \, dA_s}_{\text{path-wise}}.$$

Thm. (Dominated ~~Majorated~~ convergence). $X$ - continuous semimartingale. $k^{(n)}$ - sequence of locally bounded processes, $k^{(n)} \to 0$ pointwise. Let $\exists$ $k$ - locally bounded, such that $|k^n| \le k$. Then $(k^n \cdot X) \to 0$ in probability, uniformly on every compact interval.
Pf. For bounded variation part - usual dominated convergence

~~NB~~ ~~11~~

For local martingale part, $\exists$ $T_n$ - stopping time, then $(k^n)^{T_n} \to 0$ in $L^2(X^{T_n})$, so, by isometric property, so does $\|k^{(n)}\cdot X^{T_n}\|_2 \to 0$. Now let $n\to\infty$ ∎

Corollary. If $k$ is left-continuous, $\Delta^n$ - sequence of subdivisions, $|\Delta^n| \to 0$, then

$$\int_0^t k_s\, dX_s = P\text{-}\lim_{n\to\infty} \sum k_{t_i} (X_{t_{i+1}} - X_{t_i})$$

Pf. RHS - stochastic integral of $k^{(n)} = \sum k_{t_i} \chi_{(t_i, t_{i+1})}$, $k^{(n)} \to k$ pointwise, $|k - k^{(n)}_n| < 2\|k\|_\infty$, so, if $k$ - bounded, get everything. General $k$ - obtained by stopping times ∎

## (17) Itô's formula.

Prop. (precursor to Itô's formula, integration by parts) $X, Y$ - continuous semimartingales. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t.$$

~~Pf.~~ In particular, $X_t^2 = X_0^2 + 2\int_0^t X_s\, dX_s + \langle X, X\rangle_t$.

Pf. Enough, by polarization, to prove for $X = Y$. For a subdivision $\Delta$, we have

$$\sum (X_{t_{i+1}} - X_{t_i})^2 = X_t^2 - X_0 - 2\sum X_{t_i} (X_{t_{i+1}} - X_{t_i})$$

Now let $|\Delta| \to 0$ ∎

Ex. $B_t^2 - t = 2\int B_s\, dB_s$.

Thm. (Itô's formula) $X = (X^1, \dots, X^n)$ - continuous semimartingales, $F \in C^2: \mathbb{R}^n \to \mathbb{R}$. Then

$$F(X_t) = F(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\, d\langle X^i, X^j\rangle_s$$

Pf. By using stopping times, can assume that Range$(X)$ is bounded. Then polynomials are dense in $C^2$, so enough to prove for them. Induction on degree. Base deg $f = 0$. Step. If true for $F$, true for $x_i F$ by integration by parts! Also, both parts are linear in $F$.

~~Corollary~~ Differential notation: $dF(X_t) = \sum \frac{\partial F}{\partial x_i}(X_t)\, dX_t^i + \frac{1}{2} \sum \frac{\partial^2 F}{\partial x_i \partial x_j}(X_t)\, d\langle X^i, X^j\rangle_t$.

Corollary. Let $F: \mathbb{R}^2 \to \mathbb{C}$, $\in C^2$,

$\frac{\partial F}{\partial y} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$. Then for any continuous local martingale $M$, $F(M, \langle M, M\rangle)$ is a local martingale.

Pf. Apply Itô, the drift term disappears ∎

Examples: $f(x,y) = x^2 - y$, $f(x,y) = e^{\lambda x - \frac{\lambda^2}{2} y}$ – exponential transform.

Remark. By Fatou's lemma, positive local martingale = supermartingale

Corollary. $B$ – $d$-dim BM ($B = (B^1, B^2, \dots, B^d)$, indep).

$f \in C^2(\mathbb{R}_+ \times \mathbb{R}^d)$, then

$$M_t^f = f(t, B_t) - \int_0^t \left(\frac{1}{2}\Delta f + \frac{\partial f}{\partial t}\right)(s, B_s)\, ds$$ – local martingale. In particular, if $f$ satisfies heat equation or $f$ harmonic and time-independent, then $f(B_t)$ – local martingale.

(8) Levy Characterization Thm

Let $X = (X^1, \dots, X^d)$ be adapted $d$-dim process, $X_0 = 0$.

TFAE

1) $X$ – B.M. in $\mathbb{R}^d$.

2) $X$ – continuous local martingale and $\langle X^i, X^j\rangle_t = \delta_{ij} t$ $\forall 1 \le i, j \le d$.

3) $X$ – continuous local martingale, and $\forall f = (f_1, \dots, f_d) \in L^2(\mathbb{R}_+, \mathbb{R}^d)$, the process

$$\mathcal{E}_t^{if} = \exp\left(i \sum_k \int_0^t f_k(s)\, dX_s^k + \frac{1}{2}\sum_k \int_0^t f_k^2(s)\, ds\right).$$

is a complex martingale

Pf. 1) ⇒ 2) by definition.

2) ⇒ 3) $\mathcal{E}^{if}$ is local martingale, by ~~corollary~~ for exp (exp. transform of $M = \sum \int f_k\, dX^k$)

It is bounded, so it is a martingale.

3) ⇒ 1) Fix $\xi \in \mathbb{R}^d$, $T > 0$, and take $f = \chi_{[0,T]} \cdot \xi$.

$$\mathcal{E}_t^{if} = \exp\left(i (\xi, X_{t\wedge T}) + \frac{1}{2}|\xi|^2 (t \wedge T)\right)$$ – martingale.

so, if $A \in \mathcal{F}_s$, $s < t < T$, then
$$E\left(\chi_A \exp\left(i\langle \xi, X_t - X_s\rangle\right)\right) = P(A)\exp\left(-\frac{|\xi|^2}{2}(t-s)\right).$$
Since it is true for $\forall \xi \in \mathbb{R}^d$, the Fourier transform of $X_t - X_s$ is $\exp\left(-\frac{|\xi|^2}{2}(t-s)\right)$ and it is independent of $\mathcal{F}_s$. so it is BM ∎

Corollary. $X_t$ – continuous local martingale, $\langle X, X\rangle_t = t \Rightarrow$ $X_t$ is a BM.
(by 2)).

## 9. Time change of a process.

Def. Time change: family $(C_s)$ of stopping times $s \geq 0$, $s \to C_s$ – a.s. increasing, continuous.
$\hat{X}_s := X_{C_s}$ – time changed process.

Properties (easy): 1) $\int_{C_0}^{C_t} H_s\, dX_s = \int_0^t \chi_{\{C_u < \infty\}} H_{C_u}\, dX_{C_u} = \int_0^{\min(t, A_\infty)} H_{C_u}\, dX_{C_u}$;

2) If $C$ a.s. finite, $X$ – continuous local martingale then $\langle \hat{X}, \hat{X}\rangle = \widehat{\langle X, X\rangle}$, $\hat{X}$ is continuous.

3) If $H$ is $(\mathcal{F}_t)'$ – progressive and $\int_0^t H_s^2\, d\langle X, X\rangle_s < \infty$ a.s. $\forall t \Rightarrow \int \hat{H}_s\, d\langle \hat{X}, \hat{X}\rangle_s < \infty$ a.s. for every $t$, and $\hat{H} \cdot \hat{X} = \widehat{H \cdot X}$ ∎

Thm. (Dambis, Dubins–Schwarz). If $M$ is a $(\mathcal{F}_t, P)$ cont. local mart, $M_0 = 0$, $\langle M, M\rangle_\infty = \infty$. Set $T_t = \inf\{s : \langle M, M\rangle_s > t\}$. Then $B_t = M_{T_t}$ is a BM, $M_t = B_{\langle M, M\rangle_t}$ (Any continuous local martingale with inf. quadratic variation is a time-changed BM)!

Pf $B_t := M_{T_t}$ is continuous local martingale, $\langle B, B\rangle_t = \langle M, M\rangle_{T_t} = t$. so, by Levy, $B$ is a BM.
One fine point: $T_{\langle M, M\rangle_t}$ can be $\leq t$, but then $M_{T_{\langle M, M\rangle_t}} = M_t$ by constancy of $M$ when $\langle M, M\rangle$ does not change ∎

## 10. Complex notation, conformal invariance, and Kakutani's Thm.

Z - complex valued L.M., $Z_t = X_t + iY_t$

$Z^2 - \langle Z,Z\rangle$ - L.M.

TFAE:

1) $Z^2$ is L.M.

conformal 2) $\langle Z, Z\rangle = 0$

orthogonal 3) $\langle X,X\rangle = \langle Y,Y\rangle$, $\langle X,Y\rangle = 0$ $\quad (\langle Z,Z\rangle = \langle X,X\rangle - \langle Y,Y\rangle + 2i\langle X,Y\rangle)$

Example 2D-B.M.

complex form of Itô:

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial F}{\partial z}(Z_s)\, dZ_s + \int_0^t \frac{\partial F}{\partial \bar z_s}(Z_s)\, d\bar Z_s + \frac{1}{4}\int_0^t \Delta F(Z_t)\, d\langle Z, \bar Z\rangle_t .$$

F - harmonic $\Leftrightarrow$ F(Z) - L.M.

F - analytic $\Rightarrow$ F(Z) - conformal martingale (from Itô).

Thm Z - conformal local martingale, $Z_0 = 0 \Rightarrow Z_t = B_{\langle X,X\rangle_t}$.

Pf. It's just a restatement of Levy: apply to $X = \mathrm{Re}\, Z$, + is parametriz. the same way, and they are uncorrelated ∎

Corollary. $F(B_{t\wedge T_\Lambda}) = F(B_0) + \tilde B_{\langle X,X\rangle_t}$ if F is conformal, for esca analytic in $\Lambda$.
where $\langle X,X\rangle_t = \int_0^t |F'(B_s)|^2\, ds$. (Itô)

Remark: if F entire, one can prove that $\langle X,X\rangle_\infty = \infty$ a.s.

Thm. (Kakutani) Let $\Omega \subset \mathbb{R}^d$ ~~bounded~~, connected, $\partial\Omega$ ~~smooth~~, $f \in C(\partial\Omega)$, $B^z_t$ - d-dim BM started at $z \in \Omega$,
$T_z = \inf\{t > 0 : B^z_t \notin \Omega\}$ - stopping time.
$u(z) := E(f(B^z_{T_z}))$. Then

1) u is harmonic in $\Omega$.

2) If $x \in \partial\Omega$ is regular then $\lim_{z\to x} u(z) = f(x)$

Def. $x \in \partial\Omega$ is called regular if $\lim_{z \to x} P(|B^z_{T_z} - x| > \varepsilon) = 0 \ \forall \varepsilon > 0$. ~~$E(T_x) = 0$ (i.e. $P(\inf\{t>0 : B^x_t \notin \Omega\}) = 0$).~~ I'll prove first for cone condition!

Examples of regular pts: Will be given later (But include cone condition, any 2D with over and etc...).

Pf. 1) Let $\Sigma$ be a disc centered at $z \in \Omega$, By OST, applied to
$T^z_\Sigma = \inf\{t : B^z_t \in \Sigma\} < T^z$, we get
$u(z) = E(f(B^z_{T^z})) = E\left(E\left(f(B^{B_{T_\Sigma}}_{T})\,\middle|\, B_{T_\Sigma}\right)\right) = E(u(B_{T_\Sigma}))$.
But $B_{T_\Sigma}$ is rotationally invariant, so the last expectation is
$\int_\Sigma u(w)\, d\sigma(w)$. So u is harmonic.
$\sigma$ measure on $\Sigma$, normalized.